

# AROUND A CONJECTURE BY R. CONNELLY, E. DEMAINE, AND G. ROTE

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**ABSTRACT.** Denote by  $M(P)$  the configuration space of a planar polygonal linkage, that is, the space of all possible planar configurations modulo congruences, including configurations with self-intersections. A particular interest attracts its subset  $M^o(P) \subset M(P)$  of all configurations *without* self-intersections. R. Connelly, E. Demaine, and G. Rote proved that  $M^o(P)$  is contractible and conjectured that so is its closure  $\overline{M^o(P)}$ . We disprove this conjecture by showing that a special choice of  $P$  makes the homologies  $H_k(\overline{M^o(P)})$  non-trivial.

## 1. INTRODUCTION

An  $n$ -linkage is a sequence of positive numbers  $l_1, \dots, l_n$ . It should be interpreted as a collection of rigid bars of lengths  $l_i$  joined consecutively by revolving joints in a closed chain.

**Definition 1.1.** For a linkage  $P$ , a *configuration* in the Euclidean space  $\mathbb{R}^d$  is a sequence of points  $R = (a_1, \dots, a_n)$ ,  $a_i \in \mathbb{R}^d$  with  $l_i = |a_i, a_{i+1}|$ ,  $n+1=1$ .

The set  $M(P)$  of all such configurations modulo the action of all isometries of  $\mathbb{R}^2$  is the *configuration space of the linkage  $P$* .

It comes together with its subset  $M^o(P) \subset M(P)$  of all configurations *without* self-intersections.

In [3] R. Connelly, E. Demaine, and G. Rote proved a strengthened version of the famous carpenter's rule conjecture. Namely, using *expansive motions* they showed that  $M^o(P)$  is contractible. In the same paper they conjectured that the closure of  $M^o(P)$  is also contractible.

We disprove the conjecture by showing that not only  $\overline{M^o(P)}$  can be non-contractible, but can also have other non-trivial homologies. For this, we use a simple trick which produces non-contractible loops in  $\overline{M^o(P)}$ . To understand the trick, it suffices to look at Fig. 2.

However, authors of [3] were motivated in their study by a physical model of a linkage which allows self-touching and self-overlapping, but does not allow the edges pass one through another, as it happens in our examples. It remains an open question whether the space becomes contractible if we forbid such passes.

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*Key words and phrases.* Configuration space, planar polygonal linkage, expansive motion, carpenter's rule .

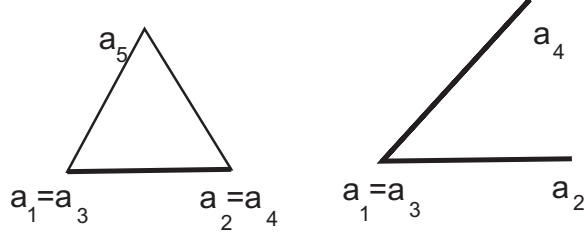


FIGURE 1. These self-touching configurations do not belong to  $\overline{M^o(P)}$

In this respect we mention two papers [1, 2] where authors treat the space of *self-touching configurations*, that is, configurations without transversal crossings. The authors equip the space by some additional structure which yields an ordering on overlapping edges. In such a space the contractible loop introduced in Section 2 becomes disconnected. In [1], it is proven that the space of self-touching configuration  $\mathcal{A}(P)$  (equipped with additional structure) is connected. To the best of our knowledge, nothing is known about contractibility of the space  $\mathcal{A}(P)$ .

However, if we forget the additional structure, the set  $\mathcal{A}(P)$  does not coincide with the set  $\overline{M^o(P)}$ : a self-touching configuration does not necessarily belong to  $\overline{M^o(P)}$ , see Fig. 1.

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## 2. A NON-CONTRACTIBLE LOOP

For a reader not acquainted with the homology theory, we start with an elementary example.

**Proposition 2.1.** *Assume that for a linkage  $P$ , we have*

$$l_1 > l_2 < l_3, \text{ and } l_1 - l_2 + l_3 < \sum_{j \neq 1,2,3} l_j.$$

*Then the space  $\overline{M^o(P)}$  contains a non-contractible loop.*

*Proof.* Consider a continuous mapping  $\alpha : M(P) \rightarrow S^1$  which maps a configuration of the linkage to the value of the oriented angle  $\angle(a_1 a_2 a_3)$ .

Next, consider a loop  $\gamma(t) : S^1 \rightarrow \overline{M^o(P)}$  which is depicted in Fig. 2. All the points of  $\gamma(S^1)$  except for exactly one (which corresponds to the configuration with overlapping edges) lie in  $M^o(P)$ . An appropriate choice of parameterization of  $\gamma$  makes the following diagram commute:

$$\begin{array}{ccc} & \overline{M^o(P)} & \\ \gamma \nearrow & & \searrow \alpha \\ S^1 & \xrightarrow{id} & S^1. \end{array}$$

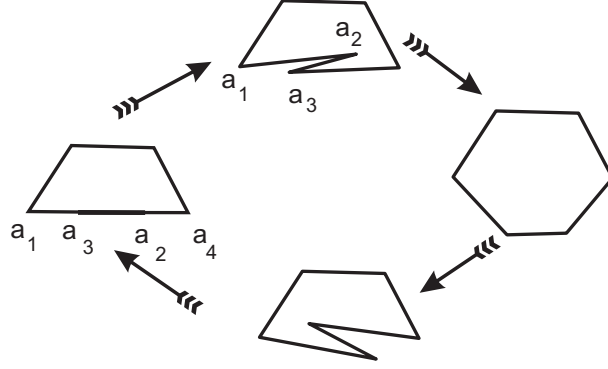


FIGURE 2. A triple fold yields a non-contractible loop

(Here and in the sequel,  $id$  denotes the identity mapping.) This means that the loop  $\gamma$  is non-contractible.  $\square$

In this respect, we conject that for any linkage  $P$ , the space

$$\overline{M^o(P)} \setminus \{\text{configurations with triple folds}\}$$

is contractible.

### 3. NON-TRIVIAL HOMOLOGIES OF THE SPACE $\overline{M^o(P)}$

**Theorem 3.1.** *For every  $m \in \mathbb{N}$ , there exists a linkage  $P$  such that for all  $k \leq n$ , all the homology groups  $H_k(\overline{M^o(P)})$  are non-trivial.*

*Proof.* We construct a polygonal linkage with  $n = 4m$  edges, combining  $m$  triple folds from the previous section, see Fig. 3.

Following the pattern of Section 2, we consider a continuous mapping

$$\alpha : \overline{M^o(P)} \rightarrow T^m = S^1 \times \dots \times S^1,$$

which maps a linkage  $P$  to the string of oriented angles

$$\alpha(P) = (\angle(a_1 a_2 a_3), \angle(a_6 a_7 a_8), \dots).$$

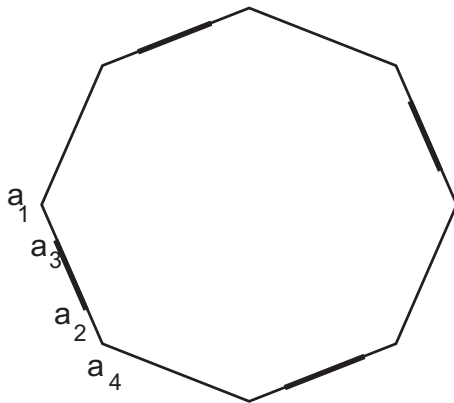
Besides, analogously to the above, we get a mapping

$$\gamma : T^m = S^1 \times \dots \times S^1 \rightarrow \overline{M^o(P)}.$$

We may freely assume that the parameterization of  $\gamma$  makes the following diagram commute:

$$\begin{array}{ccc} & \overline{M^o(P)} & \\ \gamma \nearrow & & \searrow \alpha \\ T^m & \xrightarrow{id} & T^m \end{array}$$

This immediately implies the following commutative diagram for homology groups (see [4]):

FIGURE 3.  $m$  triple folds yield  $m$  non-homological loops

$$\begin{array}{ccc}
 & H_k(\overline{M^o(P)}) & \\
 \gamma_k \nearrow & & \searrow \alpha_k \\
 H_k(T^m) & \xrightarrow{id} & H_k(T^m).
 \end{array}$$

So, the group  $H_k(T^m)$  (which is a free abelian group of rank  $\binom{m}{k}$ ) is a subgroup of  $H_k(\overline{M^o(P)})$ .  $\square$

## REFERENCES

- [1] T. Abbott, E. Demaine, B. Gassend, *A generalized carpenter's rule theorem for self-touching linkages*. arXiv:0901.1322v1
- [2] R. Connelly, E. Demaine, G. Rote, *Infinitesimally locked self-touching linkages with applications to locked trees*. In: Calvo, J., Millett, K., Rawdon, E. (eds.) *Physical Knots: Knotting, Linking, and Folding of Geometric Objects in 3-space*, pp. 287-311. American Mathematical Society, Providence (2002)
- [3] R. Connelly, E. Demaine, G. Rote, *Straightening polygonal arcs and convexifying polygonal cycles*. *Discrete Comput. Geom.* 30, No. 2, 205-239 (2003).
- [4] A. Dold, *Lectures on algebraic topology*. *Die Grundlehren der Math. Wissenschaften*, Band 200, Springer-Verlag, New York-Berlin, 1972.